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ERRORS IN COMPUTATIONS AND THE ROUNDED NUMBER[1]

By PROFESSOR HARRIS RICE,
Worcester Polytechnic Institute

I believe it is now quite generally felt among teachers of mathematics that it is their duty to give the boy and girl such mathematical instruction as may be of use to them in their daily life after leaving the high school. This feeling has given rise to some of the modern ideas as to what should be taught the boy and girl who does not go beyond the high school. Therefore they say properly that such students should be given a little trigonometry, a little differential calculus, a little integral calculus, not very much of these topics, but just enough to give the student an idea of what these subjects are and for what they are used. Such modern ideas seem to the writer very desirable. Mathematics has too long been held up as a very mysterious subject, something which the boys and girls of the high school have been very apt to dread—and if mention should be made of such a terrible subject as calculus, they would give up in despair.

The object of the present paper is not to cry down the introduction of such material into the high school curriculum but rather to urge that more emphasis be put upon some subject matter already there. I say more emphasis, but I am almost tempted to say "some mention should be made of it," for I find that in many secondary schools this topic which I have in mind is not touched upon at all. This condition exists not because the subject matter is difficult to grasp but because the rank and file of teachers are not familiar with it. Text book writers have made this subject so difficult that a great many teachers think it is way beyond them. It is my intention in the present paper to lead mathematics teachers to feel differently towards this very important part of mathematical instruction.

I have chosen for the title of my paper, "Errors in Computations and the Rounded Number." This does not mean that I shall give a discussion of the theory of errors nor that I shall discuss the question of accuracy from the differential point of view as is done in most text books. I have avoided such a discussion and also the differential notation on the grounds that such a method of attack would be of no use to the secondary school pupil. I do this not because I doubt for a moment the ability of my readers to follow such a discussion but because I have a certain objective in mind; to make this subject seem not only desirable for secondary school work but a necessary part of any secondary school course in mathematics.

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[1] A paper read before The Association of Teachers of Mathematics in New England, Boston, Mass., May 6, 1922.

Among all the various classes into which numbers have been divided since the days of the early mathematicians, let us consider for a moment the two classes known as counting and measurement numbers. A counting number is the number given in answer to the question, "How many?" In this class of numbers belongs zero and positive integers. Negative numbers and fractions have no place in this system. For instance, in answer to the question: How many were present at the meeting? we would never give such answers as—9 or 5½. Our answer would be a very definite positive number as 15, 25, 0, etc.

Perhaps you are now ready to say that a measurement number answers like questions. It counts up the number of inches or feet or whatever the unit of measurement is, you may say, and therefore I have no right to put measurement numbers into a different class from counting numbers. I admit that they do count up the number of units of measurement, but let us see if there really is not a difference after all between what I have called a counting number and what you have called a counting number.

Suppose I ask you to measure the length of a certain table, and you say after performing the task that the length is 3 ft. 7⅜ in. It appears that you have scored against me for you have given me in your answer a fraction, something which I said could not exist in my system of counting numbers. "Very well," you may say, "I inadvertently gave you my answer in that form. I might just as well have said that the unit of measure is one eighth and I find that there are 347 units in that length." This answer apparently satisfies the definition I have given for a counting number, but I still insist that there are reasons why it cannot be included in the class of counting numbers.

I can now point out the fundamental difference between a counting number and a measurement number. When you tell me that there are 15 people at the meeting, you mean that there are exactly 15, no more or no less. But when you say that the length of the table is 3 ft. $7\frac{3}{8}$ in., can you hold up your right hand and swear that your measurement is so exact that the true length is not .001 in. more or less than your 3 ft. $7\frac{3}{8}$ in.? Of course you cannot do so. What you have given represents the length of the table to the nearest eighth of an inch and that is all. You know that it is nearer 3 ft. $7\frac{3}{8}$ in. than it is 3 ft. $7\frac{2}{8}$ in. or 3 ft. $7\frac{4}{8}$ in., but beyond that you cannot go. You may select measuring sticks with finer degrees of division and measure your table to the nearest 16th, 32nd, 64th of an inch, but no matter how minute the division you can never say with absolute certainty that you have recorded the exact length of the table. Such a number is what I call a true measurement number. While it does count up, it counts up in an entirely different way than a counting number does.

All measurement numbers are thus seen to be approximate. They represent a length to the nearest fraction of a unit, the degree of approximation depending upon the care we take in measuring the length, the accuracy of our measuring instrument, and the size of the smallest unit on the instrument employed. In any case they are approximate or, as we say, represent the length in round numbers. Such numbers are therefore called rounded numbers. Any number which is used as an approximation is called a rounded number. The radius of the earth in round numbers is 4000 miles.

Since a rounded number is any number which is used as an approximation in place of the true value, the class of rounded numbers is not restricted to measurement numbers alone. While all measurement numbers are rounded numbers, all rounded numbers are not measurement numbers. Nearly all decimal fractions are rounded numbers. When .33 is used for $\frac{1}{3}$ we approximate the value of $\frac{1}{3}$. We say that $\frac{1}{3}$ in round numbers is .33. Trigonometric functions, logarithms, irrational numbers are rounded numbers. The value of π in round numbers is 22/7.

We say that the value in round numbers of $\frac{1}{3}$ to two decimal places is .33; the sine of 45 degrees to four decimal places is

.7071; the logarithm of 3 to seven decimal places is .4771213; the value of π to fifteen decimal places is 3.141592653589793.

Example 1. *Rounded Numbers.*

$$\tfrac{1}{3} = .33$$
$$\sin 45^\circ = .7071$$
$$\log 3 = .4771213$$
$$\pi = 3.14159\ 26535\ 87993$$

These are all rounded numbers given to so many decimal places. Such numbers give rise to our familiar 4, 5, 7, 10 place tables.

In each of the above cases I have given the number rounded to so many decimal places. It is much more convenient in general to speak of a number as being rounded to so many significant figures. We would then say that we have given ⅓ to two significant figures; the sine of 45 degrees to four significant figures; the logarithm of 3 to seven significant figures; the value of π to sixteen, not fifteen significant figures.

What are significant figures? The digits 1 to 9 inclusive are always significant. A zero may or may not be significant, depending upon its position in the number. If a number is composed of the digits 1 to 9 inclusive only, then the number of significant figures is determined by counting the number of digits present. If a zero occurs in the number, it is necessary first to determine whether or not it is significant. The significant digit lying on the extreme left is called the head digit as it is the most important digit in the number.

If the zero lies between two digits known to be significant, then it is significant also. For instance, (see example 2), 72.03 is a number of four significant figures; 200.24 is one of five.

Example 2. *Significant Zeroes.*

72.03   200.24

00.00241   .00241

$$\tfrac{1}{4} = .2500$$
$$\tfrac{1}{3} \neq .330$$
$$2.340$$

If the zero lies to the left of all digits known to be significant, it is not significant, and should never be used in this way except between the decimal point and the first significant digit as in the case of decimal fractions. The head digit can never be zero. Thus we should never write 00.00241, but instead .00241. Neither of these two zeroes is significant. They are simply fillers to locate the decimal point in its proper place. The number .00241 is thus a number of three significant figures.

If the zero lies to the right of all digits known to be significant, then it may or it may not be significant. If I say that $\frac{1}{4}$ to four decimal places is .2500, both zeroes are significant and we have $\frac{1}{4}$ expressed in round numbers to four significant figures; if I say that $\frac{1}{3}$ in round numbers is .330, then the zero employed is not significant, and I have expressed the value of $\frac{1}{3}$ to only two significant figures; if I have given the number 2.340, I say that I do not know whether the zero is significant or not, and so I call it a doubtful zero.

. By what means am I able to say that a right hand zero is significant, not significant, or doubtful? The answer to this question lies in the definition of a significant figure. You may have noticed that I have not yet defined what I mean by a significant figure, simply contenting myself with telling you what figures were significant in each of several cases. I have done this purposely, and I am now sure that you can define it yourself. It is simply a figure which gives some significance to the number, some added significance as regards the accuracy. You would feel much more informed in regard to the decimal value of $\frac{1}{4}$ if I should tell you that it was .25 than if I should tell you it was .2. In one case I have given the value of $\frac{1}{4}$ to two significant figures, in the other to one. In other words, a significant figure adds to the accuracy of the number. And this is the test for a right hand zero: ask yourself this question: If I knew that number more accurately would that zero disappear and some other digit take its place? If your answer is no, the zero is significant; if your answer is yes, the zero is not significant; if you cannot answer the question, the zero is doubtful.

Apply this test to each of the above cases. The value of $\frac{1}{4}$ is exactly .2500. There is no question about that, so that is why these two zeroes are significant. As a matter of fact, every zero you put on after the 5 would be significant for you could not replace them by more accurate digits. In the case of .330 standing for ⅓, we know that the zero is not correct, for ⅓ to three decimal places is .333. Hence the zero disappears when we know the number more accurately and so is not significant. In the last case, that of the number 2.340, we do not know what the number represents and so cannot say whether we should replace it by some other digit or not. Hence it is doubtful. We should never allow a zero to appear at the right of the digits 1 to 9 inclusive unless it belongs there by absolute right or unless it is put there to fix the decimal point. When we say that the radius of the earth is 4,000 miles, all three zeroes are not significant, for the radius is known to be 3,959 miles, but in giving the radius of the earth in round numbers to one significant figure these zeroes have to be put there to fix the decimal point. They determine the size of the number but not the accuracy. If on the other hand you tell me that a certain measurement yields for a result 1.500 cms., I assume that you have measured so carefully that the number of hundredths and thousandths has been determined to be zero. If not, then these zeroes should not be given, for they give a false accuracy to your result.

A very simple way to determine whether or not a zero is significant is to change the number to standard form. To change a number to standard form, place the decimal point after the head digit and offset by multiplying by the proper power of ten. (See example 3). If the zero in the original number still remains in the number when expressed in standard form, the zero is significant; if it does not so appear, it is not significant.

Example 3. *Standard Form.*

$$324.2 = 3.242 \times 10^2$$

$$.000241 = 2.41 \times 10^{-4}$$

So much by way of introduction. I have taken the time I have for this preliminary discussion for, in order that you may follow the methods of computation I am going to describe, it is necessary that you should thoroly understand what is meant by the terms *rounded numbers, significant figures,* and *standard form.*

We have seen that a rounded number is an approximation. Hence it does not represent the true value of the quantity involved. The difference between the exact value of the quantity and the value represented by the rounded number is called the absolute error. The relative error is defined to be the ratio of the absolute error to the exact value. Relative error being a ratio, is an abstract number and is often expressed in per cent. If the diagonal of a square 10 in. on a side be measured and found to be 14.1 in., the absolute error is less than .1 of an inch. The relative error is less than .71%. The error expressed as a percentage certainly means a great deal more to us.

I will now take up the fundamental processes of computation with rounded numbers. No general rules can be given save the one included in the general statement: The result of a rounded computation cannot be more accurate than the data. This means that all results should be rounded off to the same number of significant figures as is contained in the data. I shall show that this rule cannot be followed blindly, as there are exceptions. After all is said and done we employ nothing but common sense, and it is by the application of common sense alone to all of our problems that we may hope to solve our difficulties rather than by the application of any formula.

To round off a number we simply drop the superfluous figures, increasing the last digit by 1 if a 6, 7, 8, 9 is dropped; leaving it the same if a 1, 2, 3, 4 is dropped. If a 5 is dropped we want to know whether it is a 5 plus, a 5 minus, or just a 5. Suppose I have the number 23.454. See example 4. This number expressed to four significant figures would be 23.45, and I would call the 5 a 5 plus. Expressed to three significant figures the number would be 23.5. If on the other hand I have the number 28.248 and round it off to 4 significant figures, I get 28.25. This 5 I call a 5 minus, and would call this number to three significant figures 28.2. When a number ends with the digit 5 and there is no way to determine whether it is a 5 plus or a 5 minus as we did above, this rule is followed: Increase the preceding digit if it is odd; leave it unchanged if it is even.

Example 4. *Rounding Off.*

23.454	28.248
23.45	28.25
23.5	28.2

The reason for this rule is that it has been found that in the course of a large number of computations in a single problem, the number of times you increase the last digit by this rule is just about offset by the number of times you leave it unchanged.

Addition. When two or more rounded numbers are added together, it is clear that the errors will add together also. That is, the error of the sum will be the sum of the errors. Let us add the numbers 26.21, 3.26, .28, and 3.81. See example 5, (*a*).

Example 5. *Rounded Addition.*

(*a*)	26.21	.005	26.	.5	(*b*)	32.621	32.621
	3.26	.005	3.3	.05		.02841	.0284
	.28	.005	.28	.005		5.287	5.287
	3.81	.005	3.8	.05		.0002814	.0003
	33.56	.02	33.38	.6			37.9367
							37.937

Since the error in each number may be as large as .005, the error in the sum may be as large as .02. In other words, the true sum in (*a*) will lie between 33.54 and 33.58.

You have already doubtless observed that I have in my very first example disobeyed the general rule in regard to rounding off implied earlier in the paper, viz. to round off all numbers and the result to the same number of significant figures. Suppose we did follow this rule in this example. We would then have a total error of .6 possible.

If you look carefully at the columns of figures, I think no explanation is needed why the process of addition furnishes an exception to the general rule. I think you will all agree that such a rounding off process would be absurd, for while the above sum is in error at most by an amount equal to .02, the error in the sum, if the other method is used, may be as large as .6. Hence the sum would lie between 32 and 34. I say that such an error would be absurd for we know that the true sum to 3 significant figures is 33.5.

The only general rule I can give applicable to the addition of rounded numbers is the following: Round off all given numbers to be added so that there will not be more than one broken column at the right and then round off the sum so that the last figure in the sum comes in the last unbroken column. See (*b*) in example 5.

Subtraction. I shall pass over the subject of subtraction of rounded numbers with the single statement that all that was said in regard to the addition of rounded numbers applies to the subtraction of rounded numbers as well.

Multiplication. Suppose I wish to multiply the two rounded numbers a and b together. Call the error in a x, the error in b y. Then

$$(a + x)\ (b + y) = ab + ay + bx + xy$$

Since ab represents the product of the numbers a and b, the error in the product due to the errors x and y in a and b respectively must be $bx + ay + xy$. Under normal circumstances the errors x and y will be small. Hence the product xy will be very small so that it will be sufficient for our purposes to say that the error in the product is $bx + ay$.

Now suppose $a = 22.5$ and $b = 34.26$. Then x may be as large as .05 and y may be as large as .005, and

$$ay + bx = .1125 + 1.7130 = 1.8380 = 2.$$

In example 6 we have at the left the work of forming the product of these two numbers in the old-fashioned way. The third figure of our result is doubtful since the error may be as large as 2. Hence if we round off the result to three significant figures, 771, we have the product to as many significant figures as we have any right to keep. Note that this number of significant figures agrees with the smallest number of significant figures found in the original data. When two numbers are multiplied together, the result can never be accurate to a greater number of significant figures than is determined by the smallest number in the data. Hence we may save ourselves labor by rounding off both numbers to the same number of significant figures before we multiply. See the middle column of example 6. The result now is 772; to be sure, it is not 771, which we obtained before, but we have shown that the product can lie anywhere between 769 and 773.

The labor involved in such a multiplication may be still further reduced by arranging the work as is shown in the right hand column of example 6. This method may be explained thus: Round off both numbers to the same number of significant figures and then add a zero to the multiplicand, place one under the other and draw a vertical line at the right of the last figure. Multiply the multiplicand by the head digit of the multiplier, placing the first figure next to the vertical line. Then round off the multiplicand to one less significant figure and multiply the result by the second digit in the multiplier, placing the first digit of the partial product next to the line. Repeat this process until all digits of the multiplier have been used. Then add in the usual manner and round off the result to the proper number of significant figures. No attempt should be made to place the decimal point by the usual methods. Either determine where it comes by inspection or by changing to standard form.

Example 6. Rounded Multiplication.

$$\begin{array}{rrr|}
22.5 & 22.5 & 2\cancel{3}\cancel{5}\cancel{0} \\
34.26 & 34.3 & 343 \\ \hline
1350 & 675 & 6750 \\
450 & 900 & 900 \\
900 & 675 & 69 \\ \hline
675 & 77175 & 7719 \\ \cline{1-1}
 & 772. & 772. \\
770850 & & \\
771. & &
\end{array}$$

The reduction of labor is shown to better advantage perhaps in example 7. Here we multiply 24314 by .00028197. The error in the product may be as large as

$$.5 \times .00028197 + .000000005 \times 24314 = .00026$$

Hence the error comes in the fourth decimal place or, as we see from the table, in the fifth significant figure, as it should, to agree with our former statements.

Example 7. Rounded Multiplication.

$$\begin{array}{rr|}
24314 & 24\cancel{3}\cancel{1}\cancel{4}\cancel{0} \\
.00028197 & 28197 \\ \hline
170198 & 486280 \\
218826 & 194512 \\
24314 & 2431 \\
194512 & 2187 \\
48628 & 168 \\ \hline
685581858 & 685578 \\
6.8558 & 6.8558
\end{array}$$

Division. Let a and b be two rounded numbers. The question we propose to answer is: What is the error in the quotient obtained when we divide a by b? As before, let x be the greatest error in a, and y the greatest error in b. Then in place of simply a/b what we really have is $(a + x)/(b + y)$, and if we denote the error in the quotient by z we have

$$(a + x)/(b + y) = a/b + z$$

Solving for z we have

$$z = (a + x)/(b + y) - a/b = (bx - ay)/b(b + y)$$

The quantity z is thus seen to be $(bx - ay)/b(b + y)$. Those of you who are familiar with differentials recognize at once the differential form of the expression for z. Practically all text books use the differential as the error. But in my attempt to treat this subject from the point of view of the high school boy or girl I have obtained the result without reference to the differential notation.

In the expression for z, the denominator consists of b^2 and by. Since y is small compared with b, by will be small compared with b^2 and may be neglected entirely. Hence the greatest error in the quotient is approximately $(bx - ay)/b^2$ or rather $(bx + ay)/b^2$ to allow for the error in a being in the other direction.

Let us now use this formula to determine how many significant figures are allowable in the quotient as compared with the number of significant figures in the given numbers.

Let $a = 22.341$ and $b = .367148$

Then $x = .0005$ and $y = .0000005$

and $(bx + ay)/b^2 = .00019474/.13480 = .001$

Hence the error may be large enough to affect the third decimal place of the quotient. Let us divide a by b according to our usual method. See example 8. There is no use in carrying the work any further for we have seen that the third decimal place is in doubt.

What are our conclusions? The quotient cannot be relied upon beyond five significant figures. This number of significant figures agrees with the smallest number of significant figures found in either of the two given numbers. Hence the rule: When two numbers known to be rounded numbers are to be divided, equalize the number of significant figures in each by rounding off the numbers.

The labor of dividing two such numbers will be somewhat shortened if the following method is used, only when this method is used, the precaution must be taken of keeping one extra figure in the dividend when the head digit of the dividend is less than the head digit of the divisor. This must be done in order that the short method will yield a quotient containing the proper number of significant figures. The second column in example 8 illustrates this method. Note the zero added to the dividend for the reason just explained. If you compare the result obtained by this method with the one obtained earlier, you will see that the difference between the two is within the limit of the variation allowed. Hence the result is just as accurate and the labor involved is very much less when this method is used.

The method I think is self explanatory. Instead of adding zeroes to the dividend I have rounded off the divisor one digit at a time after each partial quotient. When zeroes are added to the dividend as in the old-fashioned method, there are assumed a greater number of significant figures in the dividend than are given. There is just as much of an assumption then in the long method as there is in the short method.

Example 8. *Rounded Division.*

$$\begin{array}{r|r|l}
.367148 & 22.34100000 & 60.848 \\
 & \underline{22\,02888} & \\
 & \underline{3121200} & \\
 & 2937184 & \\
 & \underline{1740160} & \\
 & 1468592 & \\
 & 281568 &
\end{array}$$

$$\begin{array}{r|r|l}
4\cancel{7} & & \\
36\cancel{7}\cancel{1}\cancel{5} & 223410 & 60849 \\
 & \underline{220290} & \\
 & 3120 & \\
 & \underline{2936} & \\
 & 184 & \\
 & \underline{148} & \\
 & 36 &
\end{array}$$

Square. Let us next turn our attention to the error in a square. Again let a be rounded, the greatest error in a being x. Then

$$(a + x)^2 = a^2 + 2ax + x^2$$

Hence the error in the square may be as large as $2ax + x^2$. Since x is small compared with a, x^2 is small compared with $2ax$, and the error in the square is usually taken to be $2ax$. Since this is simply the product of two numbers which we have already considered in detail, I will not take your time by applying this formula to a numerical case.

Square Root. Using the same notation as before, we may say

$$a + x = \text{(very nearly) } a + x + x^2/4a$$

since the last term is small compared with a. Then

$$\sqrt{a + x} = a + x/2a$$

so that the error in the square root is $x/2a$.

$$\text{Let } a = 1681. \text{ Then } x = .5 \text{ and}$$

$$x/2a = .5/82 = .005$$

Hence the square root of 1681 could be found to five significant figures.

Let us take another case. Let $a = 11.56$. Then $x = .005$ and

$$x/2a = .005/6.8 = .0005$$

Hence again we would be permitted to obtain a result containing one more significant figure than the original number contains. That the square root should contain the same number of significant figures as the original number is not at all surprising when we remember that the two numbers whose product is a number of four significant figures must contain four significant figures themselves. But for the square root to be allowed one more significant figure then the number contained in the original number I admit makes us pause and think. A moment's thought will show that such a result is reasonable, however. When we multiply we increase the error; when we take a square root we decrease the error.

Let me again call attention to my purpose in this paper. that of presenting the subject in such a way that it may seem not only desirable but also quite necessary to include this subject in the course in secondary mathematics. It is the high school boy and girl who needs these processes. Introduce significant figures and rounded numbers in connection with mensuration problems. Show your students how absurd it is to keep nine or ten figures in a result which is reliable to only a third of that number of figures. Make them see that it is very largely a matter of common sense. As a final word, let me quote the following taken from a book by Professor Ransom: "The rounding processes give results whose accuracy is as great as is permitted by the accuracy of the data, from 25% to 40% of the figuring is saved, and the false appearance of exactness is avoided."